(Ω-)Provably Δ_1 Games

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Long games

Joint work with Doug Blue.

Long games

Theorem

Suppose there is an iterable proper class inner model of ZFC *with a measurable Woodin cardinal* $\delta < \omega_1^V$ *and let* $A \subset 2^{\omega_1}$ *be* Δ_1*, provably in* ZFC*. Then* A *is determined.*

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Theorem

Suppose there is an iterable proper class inner model of ZFC *with a measurable Woodin cardinal* $\delta < \omega_1^V$ *and that the Continuum Hypothesis holds. Let* $A \subset 2^{\omega_1}$ *be* $\Delta_1(\mathbb{R})$*, provably in* ZFC*. Then* A *is determined.*

Long games

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Suppose Martin's Maximum holds. Then, there is a non-determined Σ_1 *subset of* 2^{ω_1}*.*

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Suppose Martin's Maximum holds. Then, there is a non-determined Σ_1 subset of 2^{ω_1}.

However,

Theorem (Woodin)

If there is a Woodin cardinal which is a limit of Woodin cardinals, then there is a model M of ZFC in which all definable subsets of $(2^{\omega_1})^M$ are determined.

Ω-Logic

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- Generalized logics can be obtained by restricting the class of models in which we search for validity.
- Ω-Logic is the logic of the generic multiverse.

Definition

If ϕ is a formula and T is a theory, we write $T \models_\Omega \phi$ if for all generic extensions $V[g]$ of V and all ordinals α,

$$V_\alpha^{V[g]} \models T \text{ implies } V_\alpha^{V[g]} \models \phi.$$

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Definition

A set $A \subset \mathbb{R}$ is universally Baire if for every compact Hausdorff space X and every continuous $f : X \to \mathbb{R}$, $f^{-1}[A]$ has the property of Baire (i.e., it differs from an open set by a meager set). We denote by Γ^∞ the class of all universally Baire sets.

Ω-Logic

1. If T is a tree on $\omega \times \kappa$, we write

$$p[T] = \{x \in \mathbb{R} : \forall n \in \mathbb{N}\, \exists f \in \kappa^\omega\, (x \restriction n, f \restriction n) \in T\}.$$

Theorem (Feng-Magidor-Woodin)

Let $A \subset \mathbb{R}$. The following are equivalent:

1. *A is universally Baire;*
2. *For every cardinal κ, there exist some ordinal λ and a pair of trees T, S on $\omega \times \lambda$ such that $A = p[T]$ and*

$$p[T] = \mathbb{R} \setminus p[S]$$

   in every forcing extension by a partial order of size $< \kappa$.

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Definition

Let $A \subset \mathbb{R}$ be universally Baire and M be a countable transitive model of ZFC. M is A-closed if for every partial order $P \in M$ and every V-generic $g \subset P$, we have

$$V[g] \models M[g] \cap A_g \in M[g].$$

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Definition

Let $A \subset \mathbb{R}$ be universally Baire and M be a countable transitive model of ZFC. M is strongly A-closed if for every generic extension N of M, $A \cap N \in N$.

Ω-Logic

Definition

Suppose there is a proper class of Woodin cardinals. We write

$$T \vdash_\Omega \phi$$

if there is a universally Baire set $A \subset \mathbb{R}$ such that whenever M is a (strongly) A-closed countable transitive set, we have

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Suppose there is a proper class of Woodin cardinals. Then Ω-provability is sound with respect to Ω-validity.

- The converse is known as the Ω-*conjecture*.

Ω-Logic

A useful characterization of *A*-closure:

Definition

Let $A \subset \mathbb{R}$ be universally Baire. The term relation for A is defined by

$$\tau_A^\infty = \{(p, \sigma, \gamma) : p \Vdash^V_{\mathrm{coll}(\omega,\gamma)} \sigma \in A_g\}.$$

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Lemma (Woodin)

Let $A \subset \mathbb{R}$ be universally Baire and M be a transitive model of ZFC. The following are equivalent:

1. *M is A-closed;*
2. *$\tau_A^\infty \cap b \in M$ for all $b \in M$.*

Ω-Logic

- We need a notion of *A-iterability*. Roughly, M is strongly A-iterable if it has a universally Baire iteration strategy shifting the term relation to itself.

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We can now introduce the large-cardinal notion we will use:

Definition

Let $A \subset \mathbb{R}$ be universally Baire. An A-model is a strongly A-closed, strongly A-iterable countable transitive model of ZFC with a measurable Woodin cardinal.

Long games

We can now state a generalized version of the theorem from the first slide.

Definition

Let $A \subset \mathbb{R}$ be universally Baire. A set $X \subset 2^{\omega_1}$ is Ω-*provably* $\Delta_1(A)$ if there are Σ_1-formulae in the language of set theory, ϕ and ψ, such that the following hold:

1. $X = \{f \in 2^{\omega_1} : \phi(f, A)\}$, and
2. $\mathrm{ZFC} \vdash_\Omega \forall f \in 2^{\omega_1} \, (\phi(f, A) \leftrightarrow \neg\psi(f, A))$.

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Theorem

Let X be Ω-provably $\Delta_1(A)$ for some $A \in \Gamma^\infty$. Suppose that

1. *there is a proper class of Woodin cardinals,*
2. *for all universally Baire $A \subset \mathbb{R}$, there is an A-model, and*
3. *the Continuum Hypothesis holds.*

Then X is determined.

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Long games

The following is immediate:

Corollary

Let X be $\Delta_1(A)$ in every generic extension, for some fixed $A \in \Gamma^\infty$. Suppose that

1. *there is a proper class of Woodin cardinals,*
2. *for all universally Baire $A \subset \mathbb{R}$, there is an A-model,*
3. *the Continuum Hypothesis holds,*
4. *the Ω-conjecture holds.*

Then X is determined.

Black box

The theorem is proved using the following "determinacy Black Box," which in turn is proved by closely following Neeman's proof of open/Borel determinacy for games of length ω_1.

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Theorem

Let $A \subset \mathbb{R}$ be universally Baire and $X = \{f \in 2^{\omega_1} : \phi(f, \omega_1, A)\}$ be a game of length ω_1. Suppose that

1. *for all universally Baire $B \subset \mathbb{R}$, there is a B-model, and*
2. *there is a proper class of Woodin cardinals.*

Then, there is a universally Baire $B \subset \mathbb{R}$ such that whenever M is a B-model, one of the following holds:

- *There is a strategy σ for Player I in X such that whenever $f \in V$ is by σ, f is won by Player I in a generic extension of an iterate of M that correctly computes ω_1; or*
- *Such a strategy exists for Player II.*

A digression

Definition

A set $X \subset 2^{\omega_1}$ is open if there is a formula ϕ such that

$$X = \{f \in 2^{\omega_1} : \exists\alpha < \omega_1 \, (L_{\omega_1}[f]) \models \phi(f \restriction \alpha)\}.$$

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Theorem (Neeman)

Suppose there is a countable, iterable model of ZFC *with a measurable Woodin cardinal and let* X *be open. Then* X *is determined.*

A digression

Conjecture

The following are (Ω-)equivalent over ZFC + *CH* + *proper class of Woodin limits of Woodin cardinals:*

1. *Open determinacy for games of length* ω_1*;*
2. *Provably-*$\Delta_1(\mathbb{R})$*-determinacy for games of length* ω_1*.*

Black box

We mention additional consequences of the Black Box. First, we note the following lightface version:

Theorem

Let $X = \{f \in 2^{\omega_1} : \phi(f, \mathbb{R})\}$ be a game of length ω_1. Suppose that M is a countable, iterable model with a measurable Woodin cardinal. Then, one of the following holds:

- *There is a strategy σ for Player I in X such that whenever $f \in V$ is by σ, f is won by Player I in a generic extension of an iterate of M that correctly computes ω_1; or*
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- *Such a strategy exists for Player II.*

1. The two theorems from the first slide follow easily from the Black Box.
2. Here (and before) there could be other parameters in the definition of X, but they will be shifted by the embeddings.

Applications

Theorem

Let $A \in \Gamma^\infty$. Suppose that

1. *there is a proper class of Woodin cardinals,*
2. *for all universally Baire $A \subset \mathbb{R}$, there is an A-model.*

Let

$$X = \{f \in 2^{\omega_1} : (L[f][\tau_A^\infty], f, \tau_A^\infty) \models \phi(f, A)\};$$

then X is determined.

Applications

We prove the lightface version, which is easier:

Theorem

Suppose that there is a sharp for a countable, iterable model M with a measurable Woodin cardinal. Let

$$Y = \{f \in 2^{\omega_1} : (L[f], f) \models \phi(f, \omega_1)\};$$

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Proof. Let

$$X = \{f \in 2^{\omega_1} : (L_\kappa[f], f) \models \phi(f, \omega_1)\},$$

where κ is the critical point of the topmost measure of $M^\sharp$. Let

$$X^M = \{f \in 2^{\omega_1} : (L_\kappa[f], f) \models \phi(f, \delta)\},$$

where δ is the measurable Woodin. Thus X^M is defined from δ and κ as parameters.

Applications

Applying the Black Box to X^M, we obtain one of the following:

- There is a strategy $\sigma \in V$ for Player I in X^M such that whenever $f \in V$ is by σ, f is won by Player I in a generic extension of an iterate of M that correctly computes ω_1; or
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- Such a strategy exists for Player II.

Suppose the former holds and let f be by σ. Thus, there is a model N of ZFC, an elementary embedding $j : M \to N$, and an N-generic filter g such that $f \in N[g]$ and

$$N[g] \models f \in \{f \in 2^{\omega_1} : (L_{j(\kappa)}[f], f) \models \phi(f, j(\delta))\}.$$

Applications

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- By elementarity, $j(\kappa)$ is the critical point of the topmost measure of $N^\sharp$ and, since large cardinals are preserved by small forcing, it is the critical point of a $N[g]$-measure.

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- By elementarity, $j(\kappa)$ is the critical point of the topmost measure of $N^\sharp$ and, since large cardinals are preserved by small forcing, it is the critical point of a $N[g]$-measure.
- Let N_∞ be the set-like part of the class-sized iterated ultrapower of $N[g]$ by the measure. Thus,

$$N_\infty \models f \in \{f \in 2^{\omega_1} : (L[f], f) \models \phi(f, \omega_1^V)\}.$$

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$$N_\infty \models f \in \{f \in 2^{\omega_1} : (L[f], f) \models \phi(f, \omega_1^V)\}.$$

It follows that

$$(L[f], f) \models \phi(f, \omega_1^V),$$

as desired.

Applications

We mention a final application of the Black Box. The proof follows that of Woodin's theorem from the second slide.

Theorem

Suppose there is a sharp for a model with a measurable Woodin cardinal and that the Continuum Hypothesis holds. Then, there is a class-sized transitive model M of ZFC containing $\mathbb{R}$ such that if X is a game of length ω_1 definable from $\mathbb{R}$ and real and ordinal parameters, then

$$M \models \text{"}X \text{ is determined."}$$

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$$M \models \text{“}X \text{ is determined.”}$$

Proof. We construct a model M in which all games of length ω_1 definable from $\mathbb{R}$ are determined. Using standard techniques, one can then find a model in which all games definable from ordinal and real parameters, together with $\mathbb{R}$, are determined.

Applications

- Let N be a countable model of ZFC with a measure on a cardinal greater than a measurable Woodin cardinal and let Σ be an iteration strategy for N. Let $\{s_\alpha : \alpha < \omega_1\}$ be an enumeration of $\mathbb{R}$.

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- Recursively, we define a sequence of reals $\{r_\alpha : \alpha < \omega_1\}$ such that:
  1. r_0 codes N and s_0;
  2. if $0 < \alpha$, then r_α codes s_α, together with Σ restricted to all countable iteration trees in $L[\langle r_\beta : \beta < \alpha \rangle]$.

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- Let $M = L[\langle r_\beta : \beta < \omega_1\rangle]$. The following hold:
  1. $N \in H(\omega_1)^M$,
  2. In M, there is an ω_1-iterable model with a measurable Woodin cardinal,
  3. $\mathbb{R} \subset M$ and the Continuum Hypothesis holds in M.

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  2. In M, there is an ω_1-iterable model with a measurable Woodin cardinal,
  3. $\mathbb{R} \subset M$ and the Continuum Hypothesis holds in M.
- We claim that all definable games are determined in M.

Applications

- Otherwise, let $X = \{f \in 2^{\omega_1} : \phi(f, \mathbb{R})\}$ be non-determined and define a game Y as follows:

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- Otherwise, let $X = \{f \in 2^{\omega_1} : \phi(f, \mathbb{R})\}$ be non-determined and define a game Y as follows:
- In Y, Players I and II take turns. At stage ι, Player I plays a pair of reals $(f_0(\iota), f_1(\iota))$, and Player II plays a pair $(f_2(\iota), f_3(\iota))$. This defines four functions, f_0, f_1, f_2, f_3. Let f code them all and let $f_0 \oplus f_2$ code f_0 and f_2. We set

$$Y = \{f \in 2^{\omega_1} : (L[f], f) \models \phi(f_0 \oplus f_2, \mathbb{R})\}.$$

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- By the previous theorem applied within M, Y is determined. Suppose Player I has a winning strategy σ. This induces a strategy σ^* for X by

$$\sigma^*(f_2) = f_0 \text{ if, and only if, } \exists f_1 \in M\, \sigma(f_2, \langle r_\beta : \beta < \omega_1 \rangle) = (f_0, f_1).$$

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- To every play (f_0, f_2) by σ^* corresponds a play (f_0, f_1, f_2, f_3) by σ which is won by Player I, so, by definition,

$$(f_0, f_1, f_2, f_3) \in \{f \in 2^{\omega_1} : (L[f], f) \models \phi(f_0, f_2, \mathbb{R})\}.$$

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- But $L[f] = M$, so $M \models \phi(f_0, f_2, \mathbb{R})$.
- Therefore, σ^* is a winning strategy for I. The other case is similar.

The end

Thank you.